Theory and interpretation of solar decimetric radio bursts

DOCTORAL THESIS

Ondřejov, 2002 **Miroslav Bárta**

Chapter 3

Elementary radio emission processes in the solar corona

The theory of elementary radiative processes plays a key role in determination of relations between observed data and local parameters in the source everywhere the electromagnetic radiation is used for distant diagnostics of matter. In case of radiation of coronal plasmas on radio-wave frequencies particularly three basic mechanisms are important:

- free-free thermal emission (bremsstrahlung),
- gyro-synchrotron radiation, and
- plasma emission — the process specific especially for the solar corona.

Since all these processes are closely related to properties of waves in plasmas, the elemental theory of plasma wave modes is good starting point before discussing each emission mechanism in particular.

A large amount of literature can be found on this topic – the comprehensive monograph by Melrose (1980) is roughly followed through this chapter.

3.1 Waves in plasmas

It is well known fact that in plasmas (particularly in magnetised one) a couple of wave modes can exist. This broad variety is due to a high complexity of the plasma response to electric or magnetic field perturbations. The electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields in plasmas are described by the system of Maxwell equations:

$$\begin{aligned} \text{rot }\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{div }\mathbf{E} = \frac{1}{\varepsilon_0}\rho \\ \text{rot }\mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{div }\mathbf{B} = 0 \end{aligned} \tag{3.1}$$

with $\mathbf{j}$ being the electric current density and ρ the charge density. These two quantities satisfy the charge continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div }\mathbf{j} = 0, \tag{3.2}$$

what implies directly from the set (3.1). For the purpose of formal theory of waves it is convenient to express Maxwell equations in natural basis of harmonic functions. Thus Fourier transforming the set (3.1) one obtains:

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B} \tag{3.3}$$

$$\mathbf{k} \times \mathbf{B} = -i\mu_0 j - \frac{\omega}{c^2} \cdot \mathbf{E} \tag{3.4}$$

$$\mathbf{k} \cdot \mathbf{E} = -\frac{i}{\varepsilon_0}\rho \tag{3.5}$$

$$\mathbf{k} \cdot \mathbf{B} = 0. \tag{3.6}$$

It is clear that the equation (3.6) is redundant since it follows directly from eq. (3.3), but with one exception – in the case of $\omega = 0$, i.e. in the case of static fields, the reduction of the system of equations does not apply. Thus, static fields have to be treated explicitly in further considerations. This is closely related to the well known problem of the fourth Maxwell equation (div $\mathbf{B} = 0$), which should be considered as the initial condition rather than independent relation.

From the set of three remaining equations the final wave equation in the form

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}(\mathbf{k}, \omega)) + \frac{\omega^2}{c^2}\mathbf{E}(\mathbf{k}, \omega) = -i\omega\mu_0\mathbf{j}(\mathbf{k}, \omega) \tag{3.7}$$

can be expressed, where equations (3.3) and

$$\omega\rho(\mathbf{k}, \omega) = \mathbf{k} \cdot \mathbf{j}(\mathbf{k}, \omega),$$

which is just the Fourier transform of continuity equation (3.2), should be considered as definitions of auxiliary quantities $\mathbf{B}$ and ρ in terms of basic quantities $\mathbf{E}$ and $\mathbf{j}$, respectively.

The current density $\mathbf{j}$ consists of two parts:

1. the current caused by induced motion of particles in plasmas under influence of electromagnetic field $\mathbf{j}^{ind}$
2. the extraneous current $\mathbf{j}^{ext}$

In the first approximation the induced part of current is linearly related to electric filed according to generalised Ohms law (in usual tensor notation):

$$j_i^{ind}(\mathbf{k}, \omega) = \sigma_{ij}(\mathbf{k}, \omega) \cdot E_j(\mathbf{k}, \omega) \tag{3.8}$$

where $\sigma_{ij}(\mathbf{k}, \omega)$ is the generalised conductivity tensor and usual Einsteins summation law was applied. For the formal purposes it is much more convenient to use another tensor describing the linear plasma response to electric field perturbation. The dielectric tensor $\varepsilon_{ij}(\mathbf{k}, \omega)$ is defined as:

$$\varepsilon_{ij}(\mathbf{k}, \omega) \equiv \delta_{ij} + \frac{i}{\omega\varepsilon_0} \cdot \sigma_{ij}(\mathbf{k}, \omega) \tag{3.9}$$

with δ_{ij} being the Kronecker delta (the unit tensor) and ε_0 the vacuum permittivity constant. Separating the current density into induced and extraneous parts and using Ohms law (3.8) and dielectric tensor definition (3.9) the wave equation (3.7) is re-expressed in the form:

$$\Lambda_{ij}(\mathbf{k}, \omega) \cdot E_j(\mathbf{k}, \omega) = -\frac{i}{\omega\varepsilon_0} j_i^{ext}(\mathbf{k}, \omega) \tag{3.10}$$

where the dispersion tensor $\Lambda_{ij}(\mathbf{k}, \omega)$ is defined as

$$\Lambda_{ij}(\mathbf{k}, \omega) \equiv \frac{k^2c^2}{\omega^2}\left(\frac{k_ik_j}{k^2} - \delta_{ij}\right) + \varepsilon_{ij}(\mathbf{k}, \omega). \tag{3.11}$$

The equation (3.10) represents a set of three linear equations with components of the extraneous current density $\mathbf{j}^{ext}(\mathbf{k},\omega)$ as explicit source terms.

Except of this explicit source term there is also an implicit one hidden in the dielectric tensor. The dielectric tensor can be separated into two parts – hermitian and anti-hermitian whose describes different kinds of plasma response to electric field perturbation. While the hermitian part of $\varepsilon_{ij}(\mathbf{k},\omega)$ describes time-reversible component of response[1], the anti-hermitian part causes changes of the wave energy, either positive or negative, representing such a way the effective source of waves. In case of energy decrease the damping of waves occurs, the energy increase means amplification (or negative damping/absorption) of waves.

3.1.1 The general dispersion equation of linear waves

The question arises up what is behaviour of the electric field perturbation in source-free case. Thus, one has to solve homogeneous form of the equation (3.10) with also implicit source term omitted, i.e.

$$\Lambda^h_{ij}(\mathbf{k},\omega)\cdot E_j(\mathbf{k},\omega)=0, \tag{3.12}$$

where $\Lambda^h_{ij}(\mathbf{k},\omega)$ is the hermitian part of the dispersion tensor. Solution of such a system of equations exist only if the relation

$$\Lambda(\mathbf{k},\omega)\equiv \det\Lambda^h_{ij}(\mathbf{k},\omega)=0 \tag{3.13}$$

is fulfilled. The condition (3.13) represents the general dispersion equation for linear non-damped waves in plasmas. To rewrite it to the usual form of the dispersion relation for a specific wave mode one has to express the frequency ω as a function of the wave vector $\mathbf{k}$. This is not unique operation in general however, and many branches

$$\omega^m=\omega^m(\mathbf{k}) \tag{3.14}$$

can be obtained. Each branch $\omega^m(\mathbf{k})$ represents one wave mode m.

Polarisation vectors Inserting relation (3.14) into the homogeneous equation (3.12) a solution for specific wave mode can be found. According to known rules of linear algebra the vector that solves (3.12) has to be the eigen-vector corresponding to the zero eigen-value of the tensor

$$\Lambda^h_{ij}(\mathbf{k})=\Lambda^h_{ij}\left(\mathbf{k},\omega^m(\mathbf{k})\right).$$

Such an eigen-vector is not determined uniquely since its complex amplitude is arbitrary. Therefore it is convenient to choose an unimodular complex vector $\mathbf{e}^m(\mathbf{k})$ as a representative of all solutions of the equation (3.12) for given wave mode. Such vector is called the polarisation vector and besides the dispersion relation (3.14) it is one of the basic characteristics of the specific wave mode[2].

Specific wave modes As an illustration of determination of particular wave mode and its characteristics from the general dispersion equation (3.13) one may choose well known Langmuir, transverse and ion-sound waves in plasmas without ambient magnetic field. The first thing has

[1] The corresponding part of conductivity tensor is anti-hermitian, and thus the time averaged power emitted or absorbed by plasmas $\langle\mathbf{E}\cdot\mathbf{j}\rangle$ is zero

[2] In case of two eigen-values are zeroed simultaneously one obtains two-dimensional space of solutions and the concept of polarisation vector has to be replaced introducing polarisation tensor instead (see `Melrose 1980`)

to be done is calculation of the dielectric tensor. The kinetic approach gives for unmagnetised plasmas following result (according to `Melrose 1980`):

$$\varepsilon_{ij}(\mathbf{k},\omega) = \delta_{ij} + \sum_\alpha \frac{q_\alpha^2}{\varepsilon_0\omega^2} \int \frac{(\omega - \mathbf{k}\cdot\mathbf{v})\delta_{sj} + k_s v_j}{\omega - \mathbf{k}\cdot\mathbf{v} + iO} \cdot v_i \cdot \frac{\partial f_\alpha(\mathbf{p})}{\partial p_s}\,\mathrm{d}^3\mathbf{p}, \tag{3.15}$$

the sum is performed over each particle species α and small imaginary part in the denominator indicates that correct integration path according to Landau prescription has to be used. For isotropic medium the dielectric tensor can be separated into longitudinal(ε^l) and transversal (ε^t) parts as:

$$\varepsilon_{ij}(\mathbf{k},\omega) = \varepsilon^l(k,\omega)\cdot\frac{k_i k_j}{k^2} + \varepsilon^t(k,\omega)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{3.16}$$

and explicit calculation for Maxwellian distribution function gives:

$$\begin{aligned}\varepsilon^l(k,\omega) &= 1 + \sum_\alpha \frac{1}{k^2\lambda_{D\alpha}^2}\left[1 - \phi(y_\alpha) + i\sqrt{\pi}y_\alpha\exp(-y_\alpha^2)\right]\\ \varepsilon^t(k,\omega) &= 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\left[1 - \phi(y_\alpha) + i\sqrt{\pi}y_\alpha\exp(-y_\alpha^2)\right].\end{aligned} \tag{3.17}$$

Here, $\omega_{p\alpha}$ and $\lambda_{D\alpha}$ are appropriate plasma frequencies and Debye lengths, respectively:

$$\omega_{p\alpha}^2 \equiv \frac{n_\alpha q_\alpha^2}{m_\alpha\varepsilon_0}, \qquad \lambda_{D\alpha} \equiv \frac{V_\alpha}{\omega_{p\alpha}}, \tag{3.18}$$

and the following abbreviations ($V_\alpha \equiv k_B T/m_\alpha$ designates thermal velocity of particles of species α) were used:

$$\phi(y) \equiv 2y\exp(-y^2)\int_0^y \exp(t^2)\mathrm{d}t, \qquad \mathrm{y}_\alpha \equiv \frac{\omega}{\sqrt{2}\mathrm{kV}_\alpha}.$$

Inserting the hermitian part of the dielectric tensor (i.e. retaining real parts of longitudinal and transversal components only) in the form of (3.16) into the equation (3.12) the equation

$$\left(\mathrm{Re}\left\{\varepsilon^l(k,\omega)\right\}\right)\cdot\left(n^2 - \mathrm{Re}\left\{\varepsilon^t(k,\omega)\right\}\right)^2 = 0 \tag{3.19}$$

is obtained with the refractive index n defined as

$$n \equiv \frac{ck}{\omega}.$$

The anti-hermitian part not considered for present purposes will be taken into account later in the section 3.1.2, paragraph **Absorption coefficient**.

Now, expanding the function $\phi(y)$ into series for the high-frequency limit ($y \gg 1$) and retaining only first few terms of electronic contribution to this function (the contribution of ions is reduced by factor of m_e/m_i relatively to that of electrons) the transversal part of the equation (3.19) becomes

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

or using the refractive index definition written in more familiar form

$$\omega^2(k) = \omega_{pe}^2 + c^2k^2. \tag{3.20}$$

The just derived equation (3.20) represents the dispersion equation for transversal (electromagnetic) mode.

The longitudinal part of eq. (3.19) gives two wave modes depending on the frequency limit used. For $\omega \gg kV_e$, i.e. $y_e \gg 1$ the expansion of the function ϕ yields dispersion equation

$$\omega^2(k) = \omega_{pe}^2 + 3k^2V_e^2 \tag{3.21}$$

which describes well known Langmuir waves.
On the other hand, expanding formulae for longitudinal part of the dielectric tensor in the limit

$$kV_i \ll \omega \ll kV_e$$

the ion-sound mode with the dispersion equation

$$\omega^2(k) = \frac{k^2c_s^2}{1+k^2\lambda_{De}^2} \tag{3.22}$$

is found. Here, the ion-sound wave speed c_s is defined by

$$c_s \equiv \omega_{pi} \cdot \lambda_{De}.$$

3.1.2 Energetics in the waves

The electric perturbation in plasma waves induces also the perturbation of magnetic field and, due to medium response, also variations of plasma velocity, stresses and pressure. All these perturbations raise the total amount of energy contained in plasmas and the difference over the equilibrium state can be ascribed to the waves. It is straightforward to compute the electric or magnetic field energy in waves knowing the electric field amplitude. On the other hand, mechanical energy connected with plasma motions and stresses is hard to be identified in general. Nevertheless, the total amount of energy contained in particular wave mode can be, fortunately, related to the electric field energy in this mode independently. Generally speaking, it is done rewriting the dispersion equation

$$\det\Lambda_{ij}(\mathbf{k},\omega) = 0 \tag{3.23}$$

generalised to the case of weakly damped or growing waves (the anti-hermitian part of the dielectric tensor is included now) into the form of the energy conservation law (for details see `Melrose 1980`). The explicit calculations gives for the ratio R_E^m between the total phase energy density[3] $w^m(\mathbf{k})$ and the phase energy density of the electric field $w_E^m(\mathbf{k})$ in the mode m following expression:

$$R_E^m(\mathbf{k}) \equiv \frac{w_E^m(\mathbf{k})}{w^m(\mathbf{k})} = \left(\frac{1}{\omega}\frac{\partial}{\partial\omega}\left[\omega^2\varepsilon^m(\mathbf{k},\omega)\right]\right)^{-1}_{\omega=\omega^m(\mathbf{k})}, \tag{3.24}$$

where $\varepsilon^m(\mathbf{k},\omega)$ is abbreviation for

$$\varepsilon^m(\mathbf{k},\omega) \equiv \overline{e_i^m}(\mathbf{k})e_j^m(\mathbf{k})\varepsilon_{ij}^h(\mathbf{k},\omega)$$

and the electric field (phase) energy density reads:

$$w_E^m(\mathbf{k}) = \frac{\varepsilon_0|\mathbf{E}^m(\mathbf{k})|^2}{2}\cdot\frac{(2\pi)^3}{V}. \tag{3.25}$$

[3]Phase energy density is the energy of wave mode m contained in elemental volume of phase space – i.e. energy per unit volume and unit cube of **k**-space

Energy radiated by extraneous current The extraneous current on the R.H. side of the expression (3.10) represents a source term in the wave equation. The wave energy U radiated (or absorbed) by this source is given by the work of the extraneous current against the consistent electric field of the wave, i.e.:

$$\begin{aligned} U &= -\int_{-\infty}^{+\infty} \int_V \mathbf{j}^{ext}(\mathbf{r},t) \cdot \mathbf{E}(\mathbf{r},t) \; \mathrm{d}^3\mathbf{r} \; \mathrm{d}t = \\ &= -\int_{-\infty}^{+\infty} \int \mathrm{Re} \left\{ \mathbf{j}^{ext}(\mathbf{k},\omega) \cdot \mathbf{E}(\mathbf{k},\omega) \right\} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\mathrm{d}\omega}{2\pi}, \end{aligned} \tag{3.26}$$

where the Parcevals power theorem was used. Solution of the wave equation (3.10) can be expressed as

$$E_i(\mathbf{k},\omega) = -\frac{i}{\omega\varepsilon_0} \Lambda_{ik}^{-1}(\mathbf{k},\omega) \cdot j_k^{ext}(\mathbf{k},\omega), \tag{3.27}$$

where the matrix $\Lambda_{ik}^{-1}(\mathbf{k},\omega)$ is the inversion operator to the dispersion tensor (3.11) and according to the tensor algebra rules it is written down using its co-factors (sub-determinants of transposed matrix) λ_{ik} as:

$$\Lambda_{ik}^{-1}(\mathbf{k},\omega) = \frac{\lambda_{ik}(\mathbf{k},\omega)}{\Lambda(\mathbf{k},\omega)}.$$

Now, inserting the particular solution (3.27) into the formula (3.26), the wave energy generated by the extraneous current density $\mathbf{j}^{ext}$ can be computed. Contributions to integral over ω are zero with exceptions of the poles of function in integrand. Such residues have to be treated carefully, and the integration has to be performed over the path in the complex plane according to Landau prescription. Each residue is connected with one zero of $\Lambda(\mathbf{k},\omega)$, and thus each pole represents the energy radiated in one specific wave mode. Explicit calculation gives for energy radiated by extraneous current in wave mode m the expression:

$$U^m = \int \frac{R_E^m(\mathbf{k})}{\varepsilon_0} \left| \overline{\mathbf{e}^m}(\mathbf{k}) \cdot \mathbf{j}^{ext}\left(\mathbf{k},\omega^m(\mathbf{k})\right) \right|^2 \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3},$$

where the bar over the polarisation vector $\mathbf{e}^m(\mathbf{k})$ means complex conjugation as usual. Apparently, the quantity

$$u^m(\mathbf{k}) = \frac{R_E^m(\mathbf{k})}{\varepsilon_0} \left| \overline{\mathbf{e}^m}(\mathbf{k}) \cdot \mathbf{j}^{ext}\left(\mathbf{k},\omega^m(\mathbf{k})\right) \right|^2 \tag{3.28}$$

that represents the wave energy generated by current density $\mathbf{j}^{ext}\left(\mathbf{k},\omega^m(\mathbf{k})\right)$ in the mode m per unit cube of k-space, or its time derivative – the radiated power

$$p^m(\mathbf{k}) = \frac{\mathrm{d}u^m(\mathbf{k})}{\mathrm{d}t} \tag{3.29}$$

will be more relevant ones for computation of radiation in particular emission processes, as described in the following section (3.2).

Absorption coefficient The advantage of approach used above allows one to involve also absorption of waves consistently. It can be easily done identifying the extraneous current with the implicit source term caused by anti-hermitian part $\varepsilon_{ij}^a(\mathbf{k},\omega)$ of the dielectric tensor, i.e.:

$$j_i^{ext}(\mathbf{k},\omega) = -i\varepsilon_0\omega \; \varepsilon_{ij}^a(\mathbf{k},\omega) E_j(\mathbf{k},\omega). \tag{3.30}$$

Now, the absorption coefficient

$$\gamma^m(\mathbf{k}) \equiv -\frac{1}{w^m(\mathbf{k})} \frac{\mathrm{d}w^m(\mathbf{k})}{\mathrm{d}t} = -\frac{1}{Vw^m(\mathbf{k})} \frac{\mathrm{d}u^m(\mathbf{k})}{\mathrm{d}t} \tag{3.31}$$

is computed inserting the current density (3.30) into the relation (3.28). The final result is as follows:

$$\gamma^m(\mathbf{k}) = -2i\omega^m(\mathbf{k})R_E^m(\mathbf{k})\overline{e_i^m}(\mathbf{k})e_j^m(\mathbf{k})\varepsilon_{ij}^a(\mathbf{k},\omega^m(\mathbf{k})). \tag{3.32}$$

The absorption coefficient $\gamma^m(\mathbf{k})$ may reach both positive and negative values depending on the anti-hermitian part $\varepsilon_{ij}^a(\mathbf{k},\omega)$ of the dielectric tensor. Negative values then correspond to self-generation of waves. If the anti-hermitian part $\varepsilon_{ij}^a(\mathbf{k},\omega)$ is identified with that for Maxwellian plasmas expressed by eq. (3.17) the absorption coefficient is positive and corresponding wave energy decrease is known as Landau damping.

When some wave mode experiences absorption (either positive or negative) its frequency $\omega^m(\mathbf{k})$ becomes a complex quantity what reflects changes in wave amplitude. Thus, relation between the imaginary part of frequency $\omega^m(\mathbf{k})$ and the corresponding absorption coefficient defined by eq. (3.31) should be found. Since the electric field amplitude in the wave evolves as $\mathbf{E}(\mathbf{k},t) \propto \exp[-i\omega(\mathbf{k})t]$ and $w(\mathbf{k}) \propto E^2(\mathbf{k})$ these two quantities are related simply as

$$\gamma^m(\mathbf{k}) = -2\mathrm{Im}\{\omega^m(\mathbf{k})\} \tag{3.33}$$

3.2 Specific emission mechanisms

Now, when the general theory of propagation and generation of the waves in plasmas has been reviewed, the extraneous current density and its Fourier transform has to be identified for each specific emission mechanism to calculate the energy or power radiated according to the formula (3.28).

3.2.1 Bremsstrahlung and gyro-synchrotron radiation

It is well known fact from the theory of electromagnetic field that whenever the charged particle changes the vector of its velocity the electromagnetic radiation is emitted. There are two natural kinds of such an accelerated particle motion in plasmas implying two basic emission mechanisms:

1. non-rectilinear motion of electron in the electric field of ion during electron-ion encounters in thermal plasmas – resulting radiation is called bremsstrahlung or free-free emission,

2. spiralling motion of particles (especially electrons) in the magnetic field imposed to plasmas – gyro-radiation or synchrotron radiation is the proper assignation for this radiative process in dependence on the spiralling particle energy (medium or relativistic).

Both emission processes will be now discussed further.

3.2.1.1 Bremsstrahlung

The relevant quantity that is to be found is the power $P(\omega)$ radiated in the electromagnetic mode in unit volume of plasmas per unit frequency interval. To compute this quantity one may to start with determining the energy radiated by single electron during one encounter with single ion, then calculate the power radiated by the electron during multiple (continuous) encounters with ions taking into account the encounter frequency, and finally sum over the electron distribution function.

The energy radiated during single encounter can be found from the equation (3.28) identifying the extraneous current density with that of one moving electron. If the trajectory of the electron

$$\mathbf{r} = \mathbf{r}(t)$$

is determined, the current density connected with this moving (point) charge can be expressed as

$$\mathbf{j}(\mathbf{r},t) = -e\mathbf{v}(t)\delta\left(\mathbf{r} - \mathbf{r}(t)\right)$$

and its Fourier transform is

$$\mathbf{j}(\mathbf{k},\omega) = -e\int_{-\infty}^{+\infty} \mathbf{v}(t)\exp\left[-i\left(\mathbf{k}\cdot\mathbf{r}(t) - \omega t\right)\right]\mathrm{dt}. \tag{3.34}$$

Here, $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ is the electron instantaneous velocity and $\delta(x)$ is the Dirac delta function. For the electromagnetic mode with the refractive index

$$n(\omega) \equiv \frac{ck}{\omega} \geq 1$$

in isotropic thermal plasmas (see eq. 3.20) and the non-relativistic particle ($v(t) \ll c$) the variation of the term $\mathbf{k}\cdot\mathbf{r}(t)$ is negligible comparing to the second expression ωt in the exponent. Thus, the space-dependent term contribute only by constant factor of complex unity (which can be omitted) to the Fourier transform of the extraneous current. This approximation corresponds to omitting the internal retardation inside the source in classical electromagnetic field theory (dipole approximation). Retaining the time-dependent term only in the exponent one can find for the extraneous current caused by single particle the following expression:

$$\mathbf{j}(\mathbf{k},\omega) = -e\mathbf{v}(\omega) = -\frac{ie}{\omega}\mathbf{a}(\omega), \tag{3.35}$$

with $\mathbf{a}(\omega)$ being the time Fourier transform of the particle acceleration. The total energy radiated by the accelerated charge in the electromagnetic (**T**ransversal) mode

$$U^T = \int u^T(\mathbf{k})\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}$$

can be re-expressed using the relations

$$\omega = \frac{ck}{n(\omega)}, \qquad \mathrm{d}^3\mathbf{k} = k^2\mathrm{dk}\ \mathrm{d}\Omega \tag{3.36}$$

and averaging over the solid angle Ω as the sum of contributions with frequency ω:

$$U^T = \int_0^{\infty} u^T(\omega)\mathrm{d}\omega = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2\mathrm{e}^2}{3\pi\mathrm{c}^3}\int_0^{\infty}\mathrm{n}(\omega)\left|\mathbf{a}(\omega)\right|^2\mathrm{d}\omega, \tag{3.37}$$

where formula (3.28) for energy radiated calculation with the extraneous current density (3.35) was used. For radio radiation only the distant encounters with large impact parameter b are important as the characteristic frequency radiated is about $f = v/b$. For such encounters the trajectory $\mathbf{r}(t)$ of the moving electron is only slightly departed from rectilinear motion and for the time Fourier transform of the acceleration $\mathbf{a}(t)$ in the electric field of ion

$$\mathbf{a}(t) = -\frac{e}{m_e}\frac{Z_i e}{4\pi\varepsilon_0}\frac{\mathbf{r}(t)}{|\mathbf{r}(t)|^3}$$

with the electron trajectory approximated by relation $\mathbf{r}(t) = \mathbf{b} + \mathbf{v}t$ one can write

$$\mathbf{a}(\omega) = -\frac{Z_i e^2}{m_e\ 4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{\mathbf{b}+\mathbf{v}t}{(b^2+v^2t^2)^{\frac{3}{2}}}\cdot\exp(i\omega t)\ \mathrm{dt}. \tag{3.38}$$

Here, **b** is the vector connecting the ion and the closest point on the electron straight-line path (its length is the impact parameter b), Z_i is the (fully ionised) ion proton number and $\mathbf{v}$ is the electron velocity (constant in present approximation). Computing the integral (3.38) and inserting the result into the equation (3.37) the electromagnetic energy radiated by electron during single encounter into the unit interval of frequency is expressed as:

$$u^T(\omega;b) = \frac{1}{4\pi\varepsilon_0}\frac{8n(\omega)Z_i^2e^6\omega^2}{3\pi c^3m_e^2v^4}\left[K_1^2\left(\frac{b\omega}{v}\right)+K_0^2\left(\frac{b\omega}{v}\right)\right], \tag{3.39}$$

with $K_\nu(z)$ being the MacDonald function of index ν (modified Bessel function of the second kind).

The emission of electromagnetic waves is rather continuous as electron experiences multiple encounters with many ions simultaneously. The power radiated by the single electron will be thus relevant quantity. To compute it, one has to estimate the number of collisions per unit time. Considering various impact parameters the radiated power reads:

$$P^T_{single}(\omega;v) = 2\pi n_i v\int_{b_{min}}^{\infty} u(\omega;b)\ b\ \mathrm{db}. \tag{3.40}$$

The integration has to start with the value $b_{min} > 0$ as for smaller impact parameters the present approach is not valid due to violation of straight-line approximation and/or due to need of quantum mechanic treatment of the problem.

The last step in finding the power $P^T(\omega)$ radiated by bremsstrahlung from the unit volume into unit interval of frequency consists in summing all contributions (3.40) to emitted power by single electrons with velocity v over the electron velocity distribution. Thus,

$$P^T(\omega) = \int P^T_{single}(\omega;v)f(\mathbf{v})\mathrm{d}^3\mathbf{v} \tag{3.41}$$

where for the electron distribution function $f(\mathbf{v})$ the normalisation

$$n_e = \int f(\mathbf{v})\mathrm{d}^3\mathbf{v}$$

was used. According to `Melrose 1980` the explicit computation for the Maxwellian distribution in CGS units finally gives

$$P^T(\omega) = \frac{16}{3}\frac{n(\omega)Z_i^2e^6n_in_e}{c^3m_e^2}\left(\frac{2}{\pi}\right)\frac{1}{V_e}\frac{\pi}{\sqrt{3}}\,G(T_e,\omega) \tag{3.42}$$

where the Gaunt factor

$$\frac{\pi}{\sqrt{3}}\,G(T_e,\omega) \approx \Lambda_c$$

approximately equals to the Coulomb logarithm in the classical approximation used here.

3.2.1.2 Gyro-synchrotron radiation

As already mentioned above, the gyromagnetic emission (called synchrotron radiation in case of relativistic electrons) is caused by particles spiralling in the imposed magnetic field. The presence of the field brokes the isotropy of the medium and thus the power radiated into the electromagnetic mode by unit volume into the unit frequency interval and unit solid angle, so called emissivity $\eta^T(\omega,\theta)$, will be the relevant quantity describing efficiency of the gyro-synchrotron radiation.

To estimate it, firstly the extraneous current density corresponding to particle (only electrons will be considered further) spiralling motion has to be identified. Then, using the essential relation (3.28) and the definition (3.29) the power $p^T(\mathbf{k})$ radiated in electromagnetic mode into the unit volume of $\mathbf{k}$-space by single electron can be computed. Finally, using expression (3.36) for element of $\mathbf{k}$-space in spherical coordinates the emissivity $\eta^T(\omega, \theta)$ due to single electron is computed. Moreover, for highly-relativistic particles several simplifications are allowed in the single-electron emissivity formula and in case of well-chosen, but still realistic, distribution functions the sum over particle velocity distribution can be made analytically. Thus, the expression for emissivity from the unit volume of plasmas due to synchrotron radiation is found.

The first step requires the knowledge of the particle trajectory $\mathbf{r}(t)$ that has to be inserted into the relation (3.34) to find the current density caused by single electron. When the coordinate axes are chosen appropriately (see `Melrose 1980`) the spiral orbit of the electron in the magnetic field B is described by radius-vector with the following components:

$$\mathbf{r}(t) = \Big(R\sin(\Omega t), R\cos(\Omega t), v_{||}t\Big)$$

with definitions of the gyro-frequency Ω and the Larmor radius R

$$\Omega \equiv \frac{eB}{\gamma m_e}, \qquad R \equiv \frac{v_\perp}{\Omega}$$

used; γ is the Lorentz factor. The goniometric function in the exponent of expression (3.34) is to be expanded into series with the Bessel functions $J_s(z)$ as the Fourier coefficients. Consequently, the Fourier image of the extraneous current density is

$$\mathbf{j}(\mathbf{k}, \omega) = -2\pi e \sum_{s=-\infty}^{\infty} \mathbf{V}(s, \mathbf{p}, \mathbf{k})\delta(\omega - s\Omega - k_{||}v_{||})$$

where $\mathbf{V}(s, \mathbf{p}, \mathbf{k})$ expressed in components reads

$$\mathbf{V}(s, \mathbf{p}, \mathbf{k}) \equiv \left(v_\perp \frac{s}{z} J_s(z), iv_\perp J_s'(z), v_{||} J_s(z)\right) \tag{3.43}$$

the relativistic relation for electron momentum

$$\mathbf{p} = \gamma m_e \mathbf{v}$$

was used and the argument of Bessel functions is

$$z \equiv k_\perp R = \frac{k_\perp v_\perp}{\Omega} = \frac{k_\perp p_\perp}{eB}.$$

Inserting this current density into the formula (3.28) and differencing with respect to time the power radiated by single electron into the element of $\mathbf{k}$-space in the polarisation given by the vector $\mathbf{e}^T(\mathbf{k})$ is evaluated as:

$$p^T(\mathbf{k}) = \sum_{s=-\infty}^{\infty} \frac{2\pi}{\varepsilon_0} e^2 R_E^T(\mathbf{k}) \left|\overline{\mathbf{e}^T}(\mathbf{k}) \cdot \mathbf{V}(s, \mathbf{p}, \mathbf{k})\right|^2 \delta(\omega - s\Omega - k_{||}v_{||}). \tag{3.44}$$

Now, using the element of $\mathbf{k}$-space (3.36) expressed in the spherical coordinates the emissivity due to single electron is

$$\eta^T_{single}(\omega, \theta) = \frac{e^2 n^2(\omega, \theta)\omega^2}{4\pi^2\varepsilon_0 c^3} \frac{\partial}{\partial\omega}(\omega n(\omega, \theta))\, R_E^T(\omega, \theta) \left|\overline{\mathbf{e}^T}(\theta) \cdot \mathbf{V}(s, \omega, \theta)\right|^2 \times \\ \times\delta\left[\omega(1 - n(\omega, \theta)\frac{v}{c}\cos\alpha\cos\theta) - s\Omega\right] \tag{3.45}$$

with α being the electron pitch angle ($\tan\alpha = p_\perp/p_{||}$) and where θ describes wave-vector inclination with respect to the magnetic field:

$$\cos\theta = \frac{\mathbf{k}\cdot\mathbf{B}}{|\mathbf{k}||\mathbf{B}|}.$$

Synchrotron radiation It is rather difficult to sum the single-electron emissivity (3.45) over the electron velocity distribution to find emissivity per unit volume. However, in case of relativistic energies of radiating particles several approximations can be done. For isotropic power-law in energy distribution function

$$f(E) = K\cdot E^{-a}$$

what is very common case in synchrotron radiation sources, one may for emissivity tensor write (in *CGS*, see `Melrose 1980`):

$$\eta_{ik}(\omega,\theta) = \frac{K(m_e c^2)^{1-a}}{16\pi^2 c}\sqrt{3}e^2\Omega\sin\theta\; H_{ik}(a)\left(\frac{2\omega}{3\Omega\sin\theta}\right)^{1-a}. \tag{3.46}$$

Here, the components of matrix $\mathbf{H}(a)$ are as follows:

$$H_{11} = \frac{2^{\frac{a-1}{2}}}{3(a+1)}\,\Gamma\left(\frac{3a+7}{12}\right)\Gamma\left(\frac{3a-1}{12}\right)$$

$$H_{12} = -H_{21} = \frac{-i\cot\theta}{3}\left(\frac{2\omega}{3\Omega\sin\theta}\right)^{-\frac{1}{2}} 2^{\frac{a}{2}}\,\frac{a+2}{a}\,\Gamma\left(\frac{3a+8}{12}\right)\,\Gamma\left(\frac{3a+4}{12}\right)$$

$$H_{22} = \frac{(3a+5)2^{\frac{a-3}{2}}}{3(a+1)}\,\Gamma\left(\frac{3a+7}{12}\right)\,\Gamma\left(\frac{3a-1}{12}\right),$$

a is the power-law spectral index, K the normalisation factor and $\Gamma(z)$ Euler gamma function. Using tensor generalisation of emissivity one is able to describe power radiated in arbitrary polarisation state of synchrotron radiation.

3.3 Plasma emission process

Standard radiative mechanisms – the bremsstrahlung and gyro/synchrotron radiation described above are of use also for solar corona radio emission, particularly for quiet sun radiation and slowly-variable component. Nevertheless, solar radio bursts that often consists of intense narrow-band fine structures hardly could be explained only in terms of these processes, since they have by their nature broad-band emission spectrum. Moreover, there is quantitative disagreement in values of radio flux predicted by formulae e.g. (3.42) or (3.46) using reasonable parameters of coronal plasmas and those observed during the bursts.

On the other hand, very hot and sparse coronal plasmas may, due to lack of collisions, easily be in the state of thermodynamic non-equilibrium with non-Maxwellian distribution function, particularly during solar transient events (e.g. flares or CMEs). Under such circumstances the anti-hermitian part $\varepsilon^a_{ij}(\mathbf{k},\omega)$ of the dielectric tensor (3.15) – or its counterpart for magnetised plasmas – can result to negative values of the absorption coefficient (3.32) in some range of wave-vectors for the specific wave mode m. One than says, that distribution function is unstable with respect to generation of wave mode m within some range of $\mathbf{k}$-space. The negative absorption is also often called *stimulated* or *induced* emission.

Such self-generation of waves in unstable plasmas, similar to light amplification in lasers as will be seen further, represents the basis of so called *plasma emission mechanism.* Since there are many types of distribution functions unstable to large amount of wave modes the term "plasma emission" should be regarded as generic name for all radiative processes based primarily on the negative absorption of particular wave modes.

Unfortunately, for the electromagnetic mode which only can escape from the coronal plasmas and reach Earth radiotelescopes the absorption coefficient (3.32) is always positive with one exception of so called electron-cyclotron maser radiation – see paragraph **Absorption coefficient** in the section 3.3.1.1. Thus, some mechanism of conversion between unstable plasma modes and the electromagnetic one is required. Such mechanism is available due to non-linear coupling among variations of plasma parameters (e.g. electric and magnetic field, electron density etc.) in different wave modes.

To sum up, plasma emission mechanism is generic name for class of radiative processes working usually in the following two stages:

1. the wave mode m unstable in some range of $\mathbf{k}$-space is generated due to deviation of distribution function from equilibrium Maxwellian distribution.

2. this mode m is converted via non-linear coupling into the electromagnetic one that escapes solar corona and can be detected on Earth.

Since the region of unstable waves in $\mathbf{k}$-space is usually limited to small extent and also the wave mode conversion is strongly resonant process as will be seen later, resulting radio emission is narrowband and possibly with fine structures as usually observed during solar radio bursts.

Due to mentioned similarity with radiation amplification in lasers it is convenient to adopt principle of detailed balance between emission and absorption processes used in radiative transfer elementary physics and quantitatively expressed using the Einstein coefficients. The theory built on these axioms will be in usual quantum notation briefly reviewed now.

3.3.1 Weak turbulence theory

Stimulated emission and other induced processes such as wave-particle or wave-wave scattering can be under some assumptions described consistently within the weak turbulence theory. It is based on semi-classical formalism – the particles in states with momentum $\mathbf{p}$ are described by distribution function $f(\mathbf{p})$ while the waves in mode m with wave-vector $\mathbf{k}$ is described by the occupation number $N^m(\mathbf{k})$ (number of quanta of wave mode m in state with momentum $\hbar\mathbf{k}$) defined as:

$$N^m(\mathbf{k}) = \frac{w^m(\mathbf{k})}{\hbar\omega^m(\mathbf{k})} \tag{3.47}$$

Such description brings not only the advantage of uniform treatment of various induced processes from the wave generation point of view, but also it enables consistent estimation of back-reaction of particles to wave radiation or absorption since the principle of energetic balance is imposed on microscopic level here. On the other hand, approach (3.47) to wave distribution disables correct description of coherent processes since the phase information about mode depicted by occupation number is lost. Thus, the assumption that phases of waves are unimportant – so called *random phase approximation* – plays key role in the weak turbulence theory. Coherent processes will be discussed in the next section 3.3.2, however such general theory as in case of incoherent emission has not been available yet.

One may start with subset of this general description applied to stimulated emission of waves due to unstable particle distribution function and its back-reaction to wave generation – so called *quasi-linear theory.*

3.3.1.1 Quasi-linear theory

Transferring wave generation and/or absorption processes onto microscopic level one has to use, according to quantum physics, probabilistic description of each elementary emission/absorption action. This is usually done introducing the Einstein coefficients.

Einstein coefficients Consider two states described by particle momenta $\mathbf{p}$ and $\mathbf{p}^-$. Let the total number of particles in state $\mathbf{p}$ is N_p and N_{p^-} for the state $\mathbf{p}^-$, respectively. According to quantum theory the transition of one particle between states $\mathbf{p}$ and $\mathbf{p}^-$ is accompanied by emission or absorption of quantum of waves with frequency given by condition

$$\hbar\omega = |E(\mathbf{p}) - E(\mathbf{p}^-)|. \tag{3.48}$$

Here, $E(\mathbf{p}^-)$ and $E(\mathbf{p})$ are particle energies in the states $\mathbf{p}^-$ and $\mathbf{p}$, respectively. In case of free particles the energy of the state $\mathbf{p}$ reads in non-relativistic limit

$$E(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} \tag{3.49}$$

with m being the particle mass, components of state vector $\mathbf{p}$ are simply Cartesian components of particle momentum. On the other hand, in magnetised plasmas the classical treatment is not sufficient and relativistic quantum theory gives for the eigen energies the following expression (spin of particle is ignored here, see `Melrose 1980`):

$$E(\mathbf{p}) = \sqrt{m_0^2c^4 + 2n|q|B\hbar c^2 + p_{||}^2c^2}. \tag{3.50}$$

Due to periodical motion in perpendicular direction the momentum $\mathbf{p}$ has the perpendicular component discrete – determined by the integer number n. Parallel component is continuous and corresponds to projection of momentum to the magnetic field direction, i.e.:

$$\mathbf{p} \equiv (p_\perp, p_{||}) = \left(\sqrt{2n\ \hbar\Omega\ m}, p_{||}\right) \tag{3.51}$$

where Ω is the gyrofrequency already introduced in the section 3.2.1.2.

Now suppose that $E(\mathbf{p}^-) < E(\mathbf{p})$ (see Fig. 3.1) and consider probabilities (transition rates) $\mathrm{w}_{p^-p}^{m,abs}(\mathbf{k})$, $\mathrm{w}_{pp^-}^{m,sp}(\mathbf{k})$ and $\mathrm{w}_{pp^-}^{m,ind}(\mathbf{k})$ of transitions between the states $\mathbf{p}$ and $\mathbf{p}^-$ due to absorption, spontaneous and induced emission of quantum of mode m with wave-vector $\mathbf{k}$ (referred as $(m, \mathbf{k})$ quantum further) per unit time, respectively. The rates $\mathrm{w}_{p^-p}^{m,abs}(\mathbf{k})$, $\mathrm{w}_{pp^-}^{m,sp}(\mathbf{k})$ and $\mathrm{w}_{pp^-}^{m,ind}(\mathbf{k})$ represents Einstein coefficients for transitions $\mathbf{p} \rightleftharpoons \mathbf{p}^-$. The total rate of transitions $\mathbf{p}^- \to \mathbf{p}$ due to absorption is

$$-\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t} = \mathrm{w}_{p^-p}^{m,abs}(\mathbf{k})N_{p^-}N^m(\mathbf{k}) \tag{3.52}$$

while total rate of transitions $\mathbf{p} \to \mathbf{p}^-$ as consequence of spontaneous or induced emission reads

$$\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t} = \mathrm{w}_{pp^-}^{m,sp}(\mathbf{k})N_p + \mathrm{w}_{pp^-}^{m,ind}(\mathbf{k})N_pN^m(\mathbf{k}). \tag{3.53}$$

The relations between the Einstein coefficients can be obtained in the state of thermodynamic equilibrium but it should be noted, that resulting relations are valid regardless of macroscopic state of plasma-waves system as they are fundamental characteristics of the $\mathbf{p} \rightleftharpoons \mathbf{p}^-$ transitions.

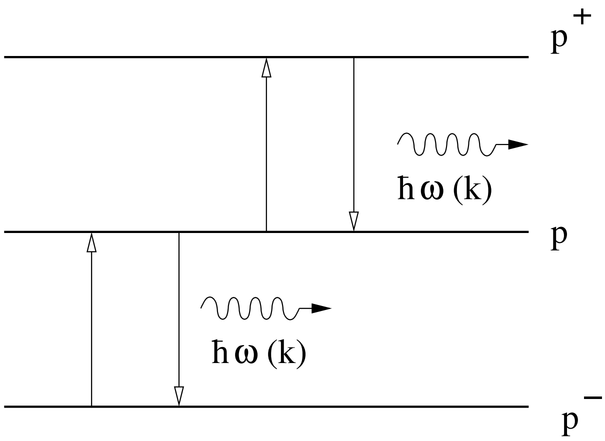

Fig. 3.1: Absorption and emission processes due to $\mathbf{p}^+ \leftrightarrow \mathbf{p}$ and $\mathbf{p} \leftrightarrow \mathbf{p}^-$ state transitions.

In the state of thermodynamic equilibrium adopted principle of detailed balance applies implying that rate of change of occupation number $N^m(\mathbf{k})$ of $(m, \mathbf{k})$ quanta due to absorption and emission processes during $\mathbf{p} \rightleftharpoons \mathbf{p}^-$ transitions together is zero. Thus combining equations (3.52) and (3.53) one obtain

$$\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t} = \mathrm{w}^{m,sp}_{pp^-}(\mathbf{k})N_p + \mathrm{w}^{m,ind}_{pp^-}(\mathbf{k})N_pN^m(\mathbf{k}) - \mathrm{w}^{m,abs}_{p^-p}(\mathbf{k})N_{p^-}N^m(\mathbf{k}) = 0. \qquad (3.54)$$

In the state of thermodynamic equilibrium the distribution of wave quanta is given by Planck law

$$N^m(\mathbf{k}) = \frac{1}{\exp\left(\frac{\hbar\omega^m(\mathbf{k})}{k_BT}\right)}.$$

Inserting the Planck law into the eq. (3.54) and taking into account that (3.54) has to apply for arbitrarily high temperature T the relation among three Einsteins coefficients is found[4] :

$$\mathrm{w}^{m,abs}_{p^-p}(\mathbf{k}) = \mathrm{w}^{m,sp}_{pp^-}(\mathbf{k}) = \mathrm{w}^{m,ind}_{pp^-}(\mathbf{k}) \equiv \mathrm{w}^{m}_{pp^-}(\mathbf{k}). \qquad (3.55)$$

Quasi-linear equations Using the relations (3.55) the rate the $(m, \mathbf{k})$ quanta are emitted at in the general (non-equilibrium) state due to all transitions that can be taken into account is (see eq. 3.54):

$$\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t} = \sum_{p,p^-} \mathrm{w}^{m}_{pp^-}(\mathbf{k}) \left[N_p + N^m(\mathbf{k})(N_p - N_{p^-})\right]. \qquad (3.56)$$

However, the actual number of possible transitions is much less than it seems from eq. (3.56) since the quantum condition

$$\mathbf{p} - \mathbf{p}^- = \hbar\mathbf{k}$$

selects only allowed ones. In particular, the transition rate $\mathrm{w}^{m}_{pp^-}(\mathbf{k})$ can be expressed as:

$$\mathrm{w}^{m}_{pp^-}(\mathbf{k}) = \mathrm{w}^m(\mathbf{p}, \mathbf{k}) \cdot \delta(\mathbf{p} - \mathbf{p}^- - \hbar\mathbf{k}). \qquad (3.57)$$

Now, one would like to change from discrete notation used hitherto to the continuous one. Thus, the number of particles N_p in the state $\mathbf{p}$ should be replaced by distribution function $f(\mathbf{p})$ and

[4] The relation somewhat differs from that used in classical radiative transfer physics because the occupation number $N^m(\mathbf{k})$ is used here instead of spectral energy density or specific intensity $I^m(\omega, \theta, \phi)$ (θ, ϕ are spherical coordinates) used in radiative transfer

double sum in the equation (3.56) by integration over $\mathbf{p}$ and $\mathbf{p}^-$. Using the expression (3.57) for the transition rate $\mathrm{w}(\mathbf{p}, \mathbf{p}^-, \mathbf{k})$, which is now re-interpreted as probability of quantum emission per unit cube of $\mathbf{k}$-space, the integration over p^- is performed trivially due to δ-function. The expression $f(\mathbf{p} - \hbar\mathbf{k})$ appeared in the result can be for $\hbar\mathbf{k} \ll \mathbf{p}$ expanded in Taylor series

$$f(\mathbf{p} \pm \hbar\mathbf{k}) = f(\mathbf{p}) \pm \hbar k_i \frac{\partial f(\mathbf{p})}{\partial p_i} + \frac{1}{2}\, \hbar^2 k_i k_j \frac{\partial^2 f(\mathbf{p})}{\partial p_i \partial p_j} + \dots$$

When only the terms that are meaningful in classical limit $\hbar \mapsto \infty$ (see the paragraph **Transition rates calculation**) are retained, the first quasi-linear equation describing wave generation (or absorption) in plasmas described by distribution function $f(\mathbf{p})$ is finally found:

$$\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t} = \int \mathrm{w}^m(\mathbf{p}, \mathbf{k}) \left(f(\mathbf{p}) + N^m(\mathbf{k}) \hbar\mathbf{k} \cdot \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} \right) \mathrm{d}^3\mathbf{p} \tag{3.58}$$

As was already mentioned, the advantage of this semi-classical approach consist besides other in possibility of homogeneous description of back-reaction of particle distribution to wave emission/absorption processes. On the microscopic level, each emission or absorption of quantum of waves is connected with transition of particle between two states. Consequently, the time change of number N_p of particles in state $\mathbf{p}$ is given by the difference between net rate the quanta $(m, \mathbf{k})$ are emitted at due to transition $(\mathbf{p}^+ = \mathbf{p} + \hbar\mathbf{k}) \to \mathbf{p}$ and net rate the quanta $(m, \mathbf{k})$ are emitted at due to transition $\mathbf{p} \to (\mathbf{p}^- = \mathbf{p} - \hbar\mathbf{k})$, i.e. (see Fig. 3.1):

$$\frac{\mathrm{d}N_p}{\mathrm{d}t} = \sum_{\mathbf{k}} \mathrm{w}^m_{p^+p}(\mathbf{k}) \left[N_{p^+} + N^m(\mathbf{k})(N_{p^+} - N_p) \right] - \sum_{\mathbf{k}} \mathrm{w}^m_{pp^-}(\mathbf{k}) \left[N_p + N^m(\mathbf{k})(N_p - N_{p^-}) \right]. \tag{3.59}$$

Transferring from the discrete notation to the continuous one again and using the Taylor expansion of the transition rate $\mathrm{w}^m(\mathbf{p}^+, \mathbf{k}) = \mathrm{w}^m(\mathbf{p}+\hbar\mathbf{k}, \mathbf{k})$ the second quasi-linear equation describing back-reaction of particles distribution to the wave emission/absorption processes reads

$$\frac{\mathrm{d}f(\mathbf{p})}{\mathrm{d}t} = \int \hbar\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}} \left[\mathrm{w}^m(\mathbf{p}, \mathbf{k}) \left(f(\mathbf{p}) + N^m(\mathbf{k})\ \hbar\mathbf{k} \cdot \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} \right) \right] \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}. \tag{3.60}$$

In the magnetised case $\mathbf{p}$ has to be interpreted in agreement with equations (3.50) and (3.51) and the wave-vector $\mathbf{k}$ consequently has components:

$$\mathbf{k} \equiv (k_\perp, k_{||}) = \left(\frac{s\Omega}{v_\perp}, k_{||} \right) \tag{3.61}$$

with s being integer number. Using these relations, the quasi-linear equations (3.58) and (3.60) can be rewritten for the case of plasmas with ambient magnetic field – the explicit calculation could be found in `Melrose 1980`.

Transition rates calculation To make equations (3.58) and (3.60) meaningful for practical computation one has to calculate the emission rate $\mathrm{w}^m(\mathbf{p}, \mathbf{k})$. It can be done when one re-interprets the power radiated $p^m(\mathbf{k})$ considered in the section 3.1.2 as continuous process to be – according to quantum physics ideas – the series of quanta emissions with emission probability per unit time $\mathrm{w}(\mathbf{p}, \mathbf{k})$, i.e.:

$$p^m(\mathbf{k}) = \hbar\omega^m(\mathbf{k})\ \mathrm{w}^m(\mathbf{p}, \mathbf{k})$$

Thus, using relations (3.28) and (3.29) the emission rate can be expressed as:

$$\mathrm{w}^m(\mathbf{p}, \mathbf{k}) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{\hbar\omega^m(\mathbf{k})} \frac{R^m_E(\mathbf{k})}{\varepsilon_0} \left| \overline{\mathbf{e}^m}(\mathbf{k}) \cdot \mathbf{j}^{ext}(\mathbf{k}, \omega^m(\mathbf{k})) \right|^2 \right) \tag{3.62}$$

In the force-free collision-less plasmas particle moves on rectilinear trajectory. Consequently, the extraneous current density in the equation (3.62) is to be identified with that given by equation (3.34) with rectilinear trajectory

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t$$

inserted. Explicit calculation gives

$$\mathrm{w}^m(\mathbf{p},\mathbf{k}) = \frac{2\pi q^2 R_E^m(\mathbf{k})}{\hbar\omega^m(\mathbf{k})\ \varepsilon_0}\,|\overline{\mathbf{e}^m}(\mathbf{k})\cdot\mathbf{v}|^2\ \delta\left(\omega^m(\mathbf{k}) - \mathbf{k}\cdot\mathbf{v}\right) \tag{3.63}$$

For the magnetised case with spiral trajectories the result can be obtained directly generalising the relation (3.44) for power radiated by single electron into transversal waves to expression valid for arbitrary mode and particle. An explicit calculation gives:

$$\mathrm{w}^m(\mathbf{p},\mathbf{k}) = \sum_{s=-\infty}^{\infty} \mathrm{w}^m(s,\mathbf{p},\mathbf{k})$$

where $\mathrm{w}^m(s,\mathbf{p},\mathbf{k})$ is abbreviation for

$$\mathrm{w}^m(s,\mathbf{p},\mathbf{k}) = \frac{2\pi q^2 R_E^m(\mathbf{k})}{\hbar\omega^m(\mathbf{k})\ \varepsilon_0}\,|\overline{\mathbf{e}^m}(\mathbf{k})\cdot\mathbf{V}(s,\mathbf{p},\mathbf{k})|^2\ \delta\left(\omega^m(\mathbf{k}) - s\Omega - k_{||}v_{||}\right) \tag{3.64}$$

with Ω being the gyrofrequency and the quantity $\mathbf{V}(s,\mathbf{p},\mathbf{k})$ defined by relation (3.43) in the section 3.2.1.2.

Absorption coefficient As was already mentioned, the first quasi-linear equation (3.58) expresses the emission or absorption of wave quanta due to medium described by distribution function. The rate of occupation number change can be separated to two parts – one independent of the occupation number itself

$$\left[\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t}\right]^{sp} = \int \mathrm{w}^m(\mathbf{p},\mathbf{k})\ f(\mathbf{p})\ \mathrm{d}^3\mathbf{p}$$

and one linearly proportional to it

$$\left[\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t}\right]^{ind} = -\gamma^m(\mathbf{k})N^m(\mathbf{k})$$

where $\gamma^m(\mathbf{k})$ reads

$$\gamma^m(\mathbf{k}) = -\int \mathrm{w}^m(\mathbf{p},\mathbf{k})\ \hbar\mathbf{k}\cdot\frac{\partial f(\mathbf{p})}{\partial\mathbf{p}}\ \mathrm{d}^3\mathbf{p}. \tag{3.65}$$

As the superscripts over each part indicate the former part describes spontaneous or thermal wave emission whereas the latter belongs to induced processes. The quantity $\gamma^m(\mathbf{k})$ is absorption coefficient by definition and its sign depend on what process prevails – whether absorption or stimulated emission of waves. In case of negative values also the term *growth rate* is often used.

It is clear from expression (3.65) that in case of positive slope of distribution function $f(\mathbf{p})$ in the direction of wave-vector $\mathbf{k}$ the absorption coefficient $\gamma^m(\mathbf{k})$ can reach negative values implying so self-amplification or instability of waves. The positive slope corresponds to inequality

$$N_{p+\hbar k} > N_p$$

in the formula (3.56), which is only discrete form of the first quasi-linear equation (3.58), and thus inverse population of energetic levels is required (in unmagnetised plasmas) for self-amplification to work. This feature of the theory of induced processes in plasmas makes it very close to, now already classical, physics of lasers as was already mentioned in the introduction to this section. Probably the most known examples of amplification of waves due to such inverse population of energetic levels in the field of plasma physics are the “Bump-in-Tail” or “Two-stream” instabilities of Langmuir waves. The positive slope of the particle distribution function is reached by energetic particle stream propagating through the thermal background plasmas in this case.

In magnetised case, the momentum $\mathbf{p}$ and the wave vector $\mathbf{k}$ has to be interpreted according to expressions (3.51) and (3.61) and, consequently, the formula (3.65 for the absorption coefficient $\gamma^m(\mathbf{k})$ rewritten for use in magnetised plasmas reads:

$$\gamma^m(\mathbf{k}) = -\sum_{s=-\infty}^{\infty} \int \mathrm{w}^m(s,\mathbf{p},\mathbf{k})\ \hbar \left(\frac{s\Omega}{v_\perp} \frac{\partial}{\partial p_\perp} + k_\| \frac{\partial}{\partial p_\|} \right) f(\mathbf{p})\ \mathrm{d}^3\mathbf{p} \tag{3.66}$$

with emission rate per s-harmonic $\mathrm{w}^m(s,\mathbf{p},\mathbf{k})$ given by the relation (3.64).

The particular importance of the formula (3.66) consist in possibility of direct amplification of electromagnetic mode in process known as *cyclotron maser.* The cyclotron maser works if: 1) conditions for negative absorption are fulfilled in some range of wave-vectors and, 2) generated electromagnetic mode can escape the solar corona. It can be shown, that the absorption coefficient (3.66) with emission rate (3.64) allows such situation under some conditions. It is unlike the unmagnetised case, where components of $\mathbf{k}$-vector is to be interpreted simply as Cartesian components of the wave vector. Then, the resonant condition contained implicitly due to δ-function in the relation (3.63) can be fulfilled only if

$$v \geq v_\varphi \tag{3.67}$$

where $v_\varphi = \omega(\mathbf{k})/k$ is the wave phase velocity. Since refractive index for electromagnetic waves $n^T(\mathbf{k}) < 1$ for all $\mathbf{k}$-vectors, negative absorption of this mode is forbidden in the case of unmagnetised plasmas as a consequence of apparent inequality

$$v < c$$

Hence, the mode conversion between waves that can satisfy the condition (3.67), and their amplification is therefore possible, and the electromagnetic ones is required for plasma emission process to work. Some mechanisms of the mode conversion will be discussed in sections 3.3.1.2 and 3.3.2.2.

Let us finish discussion of the absorption coefficient with legitimate question of relation between the coefficient (3.65) calculated within the quasi-linear theory and that determined classically in the section 3.1.2 by formula (3.32). The answer is, that both quantities are identical provided that for computation of the anti-hermitian part of dielectric tensor in the expression (3.32) the relation valid for collision-less plasmas given by equation (3.15) is used. Thus, for collision-less damping of waves the two approaches are equivalent.

3.3.1.2 Other induced processes

The indubitable advantage of the weak turbulence theory is its microscopic level approach to particles-waves system. This approach allows one to treat homogeneously not only processes where single particle and single wave quantum take part, as described by quasi-linear theory reviewed briefly in the previous section 3.3.1.1, but also multiple quanta-particle or quanta-quanta interactions as well. Among many such processes the two ones are of great importance, particularly for the second stage of plasma emission:

1. the induced scattering of waves on particles
2. the three-wave interactions

as both of them represents an efficient way of mode conversion.

Induced scattering When some wave mode propagates through plasmas the induced motion of particles (i.e. induced current) connected with this mode becomes the source of waves. In the continuum approximation for homogeneous plasmas the same wave mode with the same wave vector and frequency is reproduced only, just in agreement with the Huygens principle, and contributions to different waves mutually cancel out. Nevertheless, this is not exactly true when the discrete character of matter is taken into account. The sum of radiation from each single particle contains besides the initial wave field also component with different wave-vector/frequency or even of different mode. This component of wave field is referred as scattered wave further.

Such scattering of waves by particles is well known from classical electromagnetic field theory and as an illustrative example the Thomson scattering of light on free electrons in the solar corona can be adduced. Nevertheless, this process may have besides its spontaneous version also amplified form. It applies if the absorption coefficient (3.65) of the wave given by the envelope caused by interference between initial and scattered modes is negative. Then this envelope wave is amplified and thus the rate of the scattering process is proportional to initial and scattered wave occupation numbers, as expected for an induced process.

Both, spontaneous and induced contributions to the scattering can be described by set of kinetic equations whose derivation in the scope of the semi-classical formalism used is very similar to that of the quasi-linear equations (3.58) and (3.60) – see `Melrose 1980` for details. For scattering of the $(m, \mathbf{k})$ waves into the $(m', \mathbf{k}')$ ones (the process is usually schematicaly designated as $m \mapsto m'$ in literature) the following relations are found:

$$\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t} = \int \mathrm{w}^{mm'}(\mathbf{p},\mathbf{k},\mathbf{k}') \times \tag{3.68}$$
$$\times \left(f(\mathbf{p})[N^{m'}(\mathbf{k}') - N^m(\mathbf{k})] + N^{m'}(\mathbf{k}')N^m(\mathbf{k})\ \hbar(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} \right)\ \mathrm{d}^3\mathbf{p}\ \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}$$

and

$$\frac{\mathrm{d}N^{m'}(\mathbf{k}')}{\mathrm{d}t} = -\int \mathrm{w}^{mm'}(\mathbf{p},\mathbf{k},\mathbf{k}') \times \tag{3.69}$$
$$\times \left(f(\mathbf{p})[N^{m'}(\mathbf{k}') - N^m(\mathbf{k})] + N^{m'}(\mathbf{k}')N^m(\mathbf{k})\ \hbar(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} \right)\ \mathrm{d}^3\mathbf{p}\ \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}.$$

The first equation describes the rate of change of the occupation number of the initial wave $(m, \mathbf{k})$, similarly the second determines creation/annihilation of the scattered $(m', \mathbf{k}')$ quanta. As a consequence of scattering process the particle distribution also varies. The corresponding equation as well as the explicit expression for the scattering rate $\mathrm{w}^{mm'}(\mathbf{p},\mathbf{k},\mathbf{k}')$ can be found in `Melrose 1980`.

Three-wave interactions Another possibility of the mode conversion is given by non-linear wave-wave interactions. The physical basis of this process for simplest case of coalescence of two waves into the third is as follows: The each wave modes are independent only in the linear approximation. Nevertheless, in reality the wave propagating in plasmas where another wave is present does not see homogeneous medium but that modified by the former wave, and vice versa. As a consequence, the non-linear contribution to current appears and becomes the source of the third wave.

The process just considered above can be also described within the semi-classical formalism of the weak-turbulence theory provided the phases of each quanta are uncorrelated. This is always true in the case of broad-band distributions of waves since the coherence time for the wave mode m can be estimated as

$$\tau_c \approx \frac{1}{\Delta\omega^m(\mathbf{k})}$$

where $\Delta\omega^m(\mathbf{k})$ is the characteristic width of the mode m distribution. The coherent version of this processes is described by *strong-turbulence theory* based on the Zakharov equations and briefly reviewed in the section 3.3.2.2.

For the coalescence process of two quanta $(m', \mathbf{k}')$ and $(m'', \mathbf{k}'')$ into the third $(m, \mathbf{k})$ (and simultaneously running decay of the final quanta $(m, \mathbf{k})$ into the initial ones – schematically $m' + m'' \rightleftharpoons m$) the following set of equations can be found in the weak-turbulence approach (see `Melrose 1980`):

$$\frac{\mathrm{d}N^m(\mathbf{k})}{\mathrm{d}t} = \int \mathrm{u}^{mm'm''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \times \tag{3.70}$$
$$\times \left(N^{m'}(\mathbf{k}')N^{m''}(\mathbf{k}'') - N^m(\mathbf{k})N^{m'}(\mathbf{k}') - N^m(\mathbf{k})N^{m''}(\mathbf{k}'')\right) \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k}''}{(2\pi)^3}$$

for the coalesced wave occupation number rate of change and

$$\frac{\mathrm{d}N^{m'}(\mathbf{k}')}{\mathrm{d}t} = -\int \mathrm{u}^{mm'm''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \times \tag{3.71}$$
$$\times \left(N^{m'}(\mathbf{k}')N^{m''}(\mathbf{k}'') - N^m(\mathbf{k})N^{m'}(\mathbf{k}') - N^m(\mathbf{k})N^{m''}(\mathbf{k}'')\right) \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k}''}{(2\pi)^3}$$

and

$$\frac{\mathrm{d}N^{m''}(\mathbf{k}'')}{\mathrm{d}t} = -\int \mathrm{u}^{mm'm''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \times \tag{3.72}$$
$$\times \left(N^{m'}(\mathbf{k}')N^{m''}(\mathbf{k}'') - N^m(\mathbf{k})N^{m'}(\mathbf{k}') - N^m(\mathbf{k})N^{m''}(\mathbf{k}'')\right) \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}$$

for the initial wave quanta rates. The specific conversion rate[5] $\mathrm{u}^{mm'm''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$ can be determined expanding the current density $\mathbf{j}^{ind}(\mathbf{k}, \omega)$ induced in plasmas into series of powers of the electric field $\mathbf{E}(\mathbf{k}, \omega)$ and the resulting expression is to be found in `Melrose 1980`. Note only, that as the consequence of momentum and energy conservation on microscopic level

$$\hbar\mathbf{k}' + \hbar\mathbf{k}'' = \hbar\mathbf{k} \qquad \hbar\omega^{m'}(\mathbf{k}') + \hbar\omega^{m''}(\mathbf{k}'') = \hbar\omega^m(\mathbf{k})$$

the specific coalescence rate $\mathrm{u}^{mm'm''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$ contains delta-functions in its expression, in particular

$$\mathrm{u}^{mm'm''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \propto \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'') \times \delta[\omega^m(\mathbf{k}) - \omega^{m'}(\mathbf{k}') - \omega^{m''}(\mathbf{k}'')],$$

what indicates that three-wave interactions are strongly resonant processes.

[5]The probability that quantum from the unit cube of $\mathbf{k}$-space around the wave-vector $\mathbf{k}'$ coalesce per unit time with the quantum from the unit cube of $\mathbf{k}$-space around the wave-vector $\mathbf{k}''$ to result in the quantum from the unit cube of $\mathbf{k}$-space around the wave-vector $\mathbf{k}$

3.3.2 Coherent processes

The weak-turbulence theory just reviewed in the previous section is capable to describe homogeneously many types of particle-wave or wave-wave interactions, provided that wave field is sufficiently described by occupation numbers – i.e. that wave phases are unimportant. As was shown, such condition is fulfilled in case of broad-band wave distributions as after the coherence time τ_c the phases of waves are completely mixed. Nevertheless, sometimes the region of unstable waves in the **k**-space is so narrow, that before the phase mixing state is reached the waves have grown up substantially.

For such cases the weak-turbulence theory is inapplicable and its departure from the reality can be separated into two kinds of problems:

- the theory predicts qualitatively some process (e.g. instability) to be running, but further quantitative analysis gives wrong results – usually predicted growth rates of unstable waves are lower than in reality.
- the weak-turbulence version of coherent process does not exist at all.

Hence, processes where also wave phases are important have to be treated another way. Unfortunately, the general theory of coherent processes – as a counterpart of the weak-turbulence theory – has not been established yet. Two particular cases will be discussed in the following.

3.3.2.1 Two-stream instability

The two-stream (sometimes also, and more pertinently, designated as bump-in-tail) instability represents an example of such kind of process whose coherent as well as weak-turbulence (just quasi-linear in this case) versions exist. Usually this term is used for the situation, when thermal Maxwellian distribution (the main stream) is perturbed by low-density beam of energetic particles giving raise to unstable Langmuir waves.

Generally, the specific dispersion relation for Langmuir waves propagating through such perturbed plasmas can be found including the beam-contributed part of dielectric tensor into the general dispersion equation (3.23). For unmagnetised plasmas the separation (3.16) of the dielectric tensor into the longitudinal and transversal parts applies. The Maxwellian beam of electrons

$$f_b(\mathbf{v}) = \frac{n_b}{(\sqrt{2\pi}\Delta V_b)^3}\ \exp\left[-\frac{(\mathbf{v}-\mathbf{U}_b)^2}{2(\Delta V_b)^2}\right]$$

contributes to the longitudinal component by value (according to `Melrose 1980`):

$$\Delta\varepsilon^l(\mathbf{k},\omega) = \left(\frac{\omega_{pb}}{k\Delta V_b}\right)^2 [1-\phi(y_b)+i\sqrt{\pi}y_b\exp(-y_b^2)], \qquad (3.73)$$

where n_b is the beam electron density, $\mathbf{U}_b$ and ΔV_b the mean beam velocity and the beam velocity spread, respectively; the beam plasma frequency is defined as

$$\omega_{pb} \equiv \sqrt{\frac{n_b e^2}{m_e\varepsilon_0}},$$

the y_b is the abbreviation for

$$y_b = \frac{\omega - \mathbf{k}\cdot\mathbf{U}_b}{\sqrt{2}k\Delta V_b}$$

and the function $\phi(y)$ was introduced in the eq. (3.17). The equation (3.19) generalised for case of damped/growing waves thus gives for longitudinal (electrostatic) waves in presence of the electron beam relation

$$\varepsilon_0^l(\mathbf{k},\omega) + \Delta\varepsilon^l(\mathbf{k},\omega) = 0 \tag{3.74}$$

where $\varepsilon_0^l(\mathbf{k},\omega)$ is the background plasma contribution to the longitudinal part of the dielectric tensor given by the relation (3.17). The presence of the low-density electron beam may be regarded as a perturbation and hence the specific dispersion relation for Langmuir waves $\omega = \omega^L(\mathbf{k})$ only slightly differs from that obtained in the unperturbed case $\omega_0^L(\mathbf{k})$ and given by relation (3.21), i.e.

$$\omega^L(\mathbf{k}) = \omega_0^L(\mathbf{k}) + \Delta\omega^L(\mathbf{k}).$$

In the first order approximation the equation (3.74) can be rewritten into the form:

$$\Delta\omega^L(\mathbf{k}) \left.\frac{\partial\varepsilon_0^l(\mathbf{k},\omega)}{\partial\omega}\right|_{\omega=\omega_0^L(\mathbf{k})} + \Delta\varepsilon^l(\mathbf{k},\omega) = 0$$

from which the frequency difference $\Delta\omega^L(\mathbf{k})$ can be further expressed using the relation (3.24) as

$$\Delta\omega^L(\mathbf{k}) = -\omega_0^L(\mathbf{k}) R_E^L(\mathbf{k}) \Delta\varepsilon^l(\mathbf{k},\omega). \tag{3.75}$$

Now, when wave growth/damping is included, the function $\phi(y)$ contained in the dielectric tensor perturbation (3.73) needs to be regarded as a function of complex variable y_b. This fact brings some difficulty with expansion of the function $\phi(y)$ into series, as this expansion is not unambiguous for physically relevant case $|y_b| \gg 1$ and depends on ratio between real and imaginary part of frequency ω. In fact, only two limiting cases can be treated analytically using expansion of the function $\phi(y)$; both of them will be now briefly discussed.

1. Im[y]≫1 As a consequence of the y_b definition this limit implies the condition

$$\mathrm{Im}\{\omega\} \gg k\Delta V_b. \tag{3.76}$$

Since the bandwidth of waves unstable due to presence of the beam can be estimated by the parameter $k\Delta V_b$, the condition (3.76) means that wave growth rate is much greater then wave distribution bandwidth. In such a case the wave phases are not mixed during one growth period and become important for the process of instability. The expansion of the function $\phi(y)$ gives for this limit the following expression:

$$\Delta\varepsilon^l(\mathbf{k},\omega) \approx -\frac{\omega_{pb}^2}{(\omega - \mathbf{k}\cdot\mathbf{U}_b)^2}.$$

Inserting it into the equation (3.75) together with approximations $\omega_0^L(\mathbf{k}) \approx \omega_p$ and $R_E^L(\mathbf{k}) \approx \frac{1}{2}$ the cubic equation for the frequency difference $\Delta\omega^L(\mathbf{k})$ is obtained

$$\Delta\omega^L(\mathbf{k}) = \frac{1}{2}\,\frac{\omega_p\omega_{pb}^2}{\left(\Delta\omega^L(\mathbf{k}) + \omega_p - \mathbf{k}\cdot\mathbf{U}_b\right)^2}$$

whose solution (one of three) reads

$$\Delta\omega^L(\mathbf{k}) = \left(\frac{n_b}{n}\right)^{1/3} (\alpha + i\beta)\ \omega_p \tag{3.77}$$

with real constant α and β of order of unity (detailed result can be found in Melrose 1980). Hence, the growth rate of the coherent version of the two-stream instability can be estimated as

$$\gamma^L(\mathbf{k}) \approx \left(\frac{n_b}{n}\right)^{1/3} \omega_p$$

and it applies in the low-velocity-spread limit

$$\left(\frac{n_b}{n}\right)^{1/3} \omega_p \gg k\Delta V_b.$$

2. Re[y]$\gg$ 1 In this limit the expansion of $\phi(y)$ in the first order approximation gives:

$$\Delta\varepsilon^l(\mathbf{k},\omega) \approx -\frac{\omega_{pb}^2}{(\omega_0^L(\mathbf{k}) - \mathbf{k}\cdot\mathbf{U}_b)^2} + \frac{i\omega_{pb}^2}{(k\Delta V_b)^2}\sqrt{\pi}\; y_b \exp(-y_b^2). \tag{3.78}$$

Since imaginary part is small now due to the exponential factor $\exp(-y_b^2)$ one could expect randomisation of wave phases in times less then one growth period and, thus applicability of the weak-turbulence theory. And really, inserting the expansion (3.78) into the relation (3.75) and using obvious simplifications $\omega_0^L(\mathbf{k}) \approx \omega_p$, $R_E^L(\mathbf{k}) \approx \frac{1}{2}$ the imaginary part of $\Delta\omega^L(\mathbf{k})$ can be expressed as

$$\mathrm{Im}\{\Delta\omega^L(\mathbf{k})\} = \frac{\omega_p\omega_{pb}^2}{(\sqrt{2}k\Delta V_b)^3}\sqrt{\pi}\;(\mathbf{k}\cdot\mathbf{U}_b - \omega_p)\exp\left(-\frac{(\mathbf{k}\cdot\mathbf{U}_b - \omega_p)^2}{(\sqrt{2}k\Delta V_b)^2}\right).$$

This result can be directly compared with that obtained from quasi-linear theory using say equation (3.65). Both results are found to be identical when relation (3.33) is used.

Finally, let us note that terminology on this topic has not been unified yet and various terms designating the coherent and incoherent (quasi-linear) version of the two-stream (bump-in-tail) instability are used in the literature. The coherent instability is also frequently termed as *reactive* or, particularly in Russian literature, *of fluid type*, while for its quasi-linear version the designations *resistive* or *of kinetic type* are used as well.

3.3.2.2 Strong wave turbulence

Strong wave turbulence is generic term for non-linear wave-wave interactions that can not be sufficiently described within the weak-turbulence theory just due to great importance of wave phases for processes involved. The first description of coherent wave-wave interactions is that by Zakharov (1972) who treated the non-linear interaction between Langmuir and ion-sound waves. His approach was roughly as follows:

Firstly, let us consider linear Langmuir and ion-sound waves in homogeneous plasmas. The time evolution of plasma parameters variations in these waves can be derived most simply within the plasma two-fluid theory or alternatively they can be guessed Fourier transforming the dispersion relations (3.21) and (3.22) for relevant waves into the coordinate space. Hence, the electric field variation in Langmuir waves is governed by equation

$$\frac{\partial^2\mathbf{E}}{\partial t^2} - 3V_e^2\triangle\mathbf{E} + \omega_{pe}^2\mathbf{E} = 0 \tag{3.79}$$

and similarly the electron density variation n in the ion-sound waves fulfils (for wavelengths $\lambda \ll \lambda_{De}$) relation

$$\frac{\partial^2 n}{\partial t^2} - c_s^2\Delta n = 0. \tag{3.80}$$

Now suppose that both wave modes propagate through plasma simultaneously. Due to ion-sound wave the electron density is now distributed non-uniformly and as a consequence of the plasma frequency definition (3.18) the last term $\omega_{pe}^2\mathbf{E}$ in the eq. (3.79) depends explicitly on time and space. Hence, the equation (3.79) can be rewritten in the form

$$\frac{\partial^2\mathbf{E}}{\partial t^2} - 3V_e^2\triangle\mathbf{E} + \omega_{pe}^2\mathbf{E} = -\omega_{pe}^2\frac{n(\mathbf{r},t)}{n_0}\mathbf{E} \tag{3.81}$$

where the plasma frequency ω_{pe} is now re-interpreted as that connected with the background average density n_0. Equation (3.81) describes Langmuir wave electric field evolution under the influence of ion-sound density perturbation. The effect of density distribution can be estimated qualitatively even without solving it by analogy with the Schrödinger wave equation describing an electron inside the crystal lattice (c.f. equation 3.84). Identifying the total density n_0+n with crystal single-electron potential one finds, that the Langmuir electric field tends to concentrate itself in the density holes, similarly as electron probability density in the crystal is high in places of low potential (in the vicinity of ions locations).

On the other hand, non-homogeneous (averaged over wavelength) electric field influences density distribution due to non-linear ponderomotive force F_{NL} whose volume density is (e.g. Chen 1984):

$$\mathbf{f}_{NL} = -\frac{\omega_{pe}^2}{\omega^2}\,\mathbf{grad}\,\frac{\varepsilon_0\langle E^2\rangle}{2}. \tag{3.82}$$

where $\langle\ \rangle$ denotes the fast-time (on scales of several plasma period) averaging. As a consequence, a source term has to appear on the R.H. side of equation (3.80), i.e.

$$\frac{\partial^2 n}{\partial t^2} - c_s^2\triangle n = \frac{1}{m_i}\,\mathrm{div}\ \mathbf{f}_{NL}. \tag{3.83}$$

Since changes of electric field amplitude and ion-sound density variations are slow in comparison with plasma frequency it is convenient to separate the instantaneous Langmuir electric field time evolution into the fast (on plasma frequency) variations and the slowly varying complex amplitude

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\left[\mathcal{E}(\mathbf{r},t)\cdot\exp(-i\omega_{pe}t) + \overline{\mathcal{E}}(\mathbf{r},t)\cdot\exp(+i\omega_{pe}t)\right]$$

Using this separation and relation (3.82) for ponderomotive force, further omitting the second derivative of slowly changing complex amplitude $\mathcal{E}(\mathbf{r},t)$ the equations (3.81) and (3.83) can be rewritten in the form (see e.g. Zakharov 1972, Robinson 1997):

$$i\frac{\partial\mathcal{E}}{\partial t} + \frac{3V_e^2}{2\omega_{pe}}\triangle\mathcal{E} = \omega_{pe}\frac{n}{2n_0}\mathcal{E} \tag{3.84}$$

$$\frac{\partial^2 n}{\partial t^2} - c_s^2\triangle n = \frac{\varepsilon_0}{4m_i}\ \triangle|\mathcal{E}|^2. \tag{3.85}$$

The relations (3.84, 3.85) are known as set of Zakharov equations and describe coherent non-linear interactions of Langmuir and ion-sound waves.

Since Zakharov early work (Zakharov 1972) further generalisations of theses equations were made. The most natural one is including the electromagnetic terms into the eq. (3.81) – see e.g. Zakharov et al. 1985. The second Zakharov equation (3.85) was modified as well since in its original fluid-theory formulation was not applicable in some (particularly short-wavelength) limit – this drawback was solved using kinetic approach. Both generalised equations are explicitly written down (see equations 5.1) and used extensively further in the Chapter 5.

Let us now turn to particular non-linear wave processes described by equations (3.84, 3.85) – or their generalised version (5.1) – and known also as *parametric instabilities*.

Three-wave interactions Among three wave interactions belong coalescence of two waves into the third or decay of one pump wave into two daughter modes. Generally both these processes can be schematically written as

$$m' + m'' \rightleftharpoons m$$

where symbols m, m' and m'' may be one of L, S or T for **L**angmuir, ion-**S**ound or **T**ransversal (electromagnetic) modes if the generalised Zakharov equations (5.1) are used. The processes described here represent coherent – and therefore more efficient – version of three-wave interactions discussed in the section 3.3.1.2. In that section also the physical mechanism of the coalescence process was described. Explanation of parametric decay can be seen more clearly just in its coherent variant:

Let us suppose plane Langmuir wave (for instance) propagating through plasmas where also plane ion-sound wave is present. The density pattern formed by the ion-sound wave represents – according to the first Zakharov equation – a system of semi-reflecting mirrors for the Langmuir wave. Hence, very similarly to the Bragg reflections of X-rays in crystalline materials – but, with the Doppler shift (and/or mode change) given by motion of density pattern – the Langmuir wave is partly reflected. On the other hand, the total electric field given by the sum of both incident and reflected waves causes by means of ponderomotive force (3.82) amplification of the ion-sound wave. This amplification further leads to more efficient reflections of parent Langmuir wave and thus the positive feedback is established. The instability grow until non-linear effects of higher order will take place.

For use in solar coherent radio emission theory particularly following processes are most often considered:

$$L \to L' + S, \quad L' + S \to T$$

or

$$L \to T + S$$

for radiation on the frequency $\approx \omega_{pe}$ and

$$L' + L \to T$$

for the second harmonic ($\approx 2\omega_{pe}$) emission.

Modulational instabilities Unlike the three-wave processes discussed above the modulational instability has no weak-turbulence counterpart since coherence of waves is essential in this process. There are four waves taking part in the instability and physical mechanism is as follows:

Suppose plane Langmuir wave again propagating, for simplicity, in direction parallel to that of weak plane ion-sound wave. Due to the first Zakharov equation the Langmuir electric field tends to concentrate near density minima of ion-sound wave. Hence, the electric field of parent Langmuir wave becomes slightly modulated:

$$E^L(x) = E_0 \cos(kx) \cdot (1 + m\cos(Kx))$$

where k and K are Langmuir and ion-sound wave wave-numbers, respectively and m is small parameter of order $m \approx n/n_0$ describing modulational effect of density pattern onto Langmuir wave. The essence of the instability consist in the simple mathematical relation

$$\cos(kx)\cos(Kx) = \frac{1}{2}\ (\cos[(k+K)x] + \cos[(k-K)x])$$

indicating that side-band Langmuir waves with wave-numbers $k + K$ and $k - K$ are excited as well. As in the previous case, the ponderomotive force given by the total electric field by all three Langmuir waves causes increase of density variations what on the other hand makes the modulation deeper. Finally let us note, that in case of non-parallel mutual propagation of parent Langmuir wave and ion-sound density variation the growing side-band(s) may be radiated in the electromagnetic mode (hybrid instability) giving such also an effective way of plasma wave conversion into the radio radiation.

Both three-wave and four-wave interactions with respect to possible applications to solar flares radio radiation are further studied in the Chapter 5, section 5.1.